Multilateralism, Bilateralism and Regime Design

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Supporting Material: Proofs

Competitive equilibrium

Each member $i \in [1, N]$ chooses her level of $s_i \geq 0$ to maximize her utility $u_i = a_i \sqrt{s_i} - \rho \sum_{j \neq i} s_j - cs_i$, a function that is twice-differentiable and concave. Assuming $\lambda_i \geq 0$ to be the Lagrangian parameters, the optimal level of s_i, $s_i^{\#}$, satisfies the necessary and sufficient first-order conditions $a_i \frac{1}{2} s_i^{-\frac{1}{2}} - c + \lambda_i = 0$ and the Kuhn-Tucker conditions $s_i \lambda_i = 0$ for any $i \in [1, N]$, thus forming a system of $2N$ equations and $2N$ variables (s_i and λ_i). There is no solution possible in which, for any member $i \in [1, N]$, $\lambda_i > 0$, because it would imply $s_i^{\#} = 0$, making the corresponding first-order condition indeterminate. Therefore, the only possible determinate solution has $\lambda_i = 0$ and $s_i^{\#} = \left(\frac{a_i}{2c}\right)^2$ for all $i \in [1, N]$.

Social optimum

In any Pareto optimal allocation, the optimal level of s_i, s_i°, must maximize the joint surplus of the N members and so must solve $\max\limits_{s_i \geq 0, i \in [1,N]} \sum_{i=1}^{N} \left(a_i \sqrt{s_i} - cs_i\right) - \sum_{i=1}^{N} \rho \sum_{j \neq i} s_j$. This problem gives the necessary and sufficient first-order conditions $a_i \frac{1}{2} s_i^{-\frac{1}{2}} - c - (N-1)\rho + \gamma_i = 0$, with $\gamma_i \geq 0$ the Lagrangian parameters, and the Kuhn-Tucker conditions $s_i \gamma_i = 0$ for all $i \in [1, N]$. The problem is solved like the precedent, yielding interior solution $s_i^{\circ} = \left(\frac{a_i}{2(c+\rho(N-1))}\right)^2$ for all $i \in [1, N]$.

Solving program P

The subsidy rate. We start by determining the optimal subsidy rate, t^*. The rate must satisfy two conditions: first, it must be large enough to entice each member to abandon the competitive equilibrium for the social optimum; second, it must be high enough to deter any member from defecting to the competitive equilibrium while holding constant the optimal activity of other members. To meet the first condition, t must make the equilibrium activity under the socially optimal equilibrium at least equal to the equilibrium activity under the competitive equilibrium. Comparing the first-order conditions for each equilibrium (see above), it is straightforward to see that the condition for the optimal equilibrium is the same as that for the competitive equilibrium minus expression $(N-1)\rho$. Therefore, $t^* \geq (N-1)\rho$.

To meet the second condition, the incentive constraint in program P must be met for $s_i^* = s_i^\circ$. This means that $a_i\sqrt{s_i^\circ} - \rho\sum_{j\neq i} s_j^\circ - cs_i^\circ + t\left(s_i^\# - s_i^\circ\right) \geq a_i\sqrt{s_i^\#} - \rho\sum_{j\neq i} s_j^\circ - cs_i^\#$. Substituting the values of $s_i^\#$ and s_i° into the constraint yields $t^* \geq \frac{c(N-1)\rho}{2c+(N-1)\rho}$. Since the right hand side term is smaller than $(N-1)\rho$, it follows that this second constraint is not binding, only the first is, and thus $t^* = (N-1)\rho$.

Convexity. To show that program P is convex with respect to x and thus has a fixed-point solution, one needs to show that the founder's utility function, in which we have substituted the values for $s_i^\#$, s_i°, and t^*, is concave with respect to variables x and y. Concavity requires that for any pair of distinct points (x_1, y_1) and (x_2, y_2) in the domain of U_P, and for $0 < \theta < 1$, the following weak inequality holds: $\theta U_P(x_1, y_1) + (1-\theta)U_P(x_2, y_2) \leq U_P(\theta(x_1, y_1) + (1-\theta)(x_2, y_2))$. Developing U_P and rearranging yields $U_P = Ax^3 + Bx^2 + Cx + Dy^3 + Ey^2 + Fy + G$ with $A = -\frac{1}{6}R$, $B = \frac{1}{8}R$, $C = T + \frac{1}{24}R$, $D = -\frac{1}{12}R$, $E = -B$, $F = V - C$, $G = -T$, and $R = \rho^2(N-1)^2 a^2 \frac{2c+\rho(N-1)}{c^2(c+\rho(N-1))^2}$.

This and all subsequent calculations use the functional form for a member's marginal gain $a_i = ai$.

Concavity thus requires $\theta\left(Ax_1^3 + Bx_1^2 + Cx_1 + Dy_1^3 + Ey_1^2 + Fy_1 + G\right) + (1-\theta)$ $\left(Ax_2^3 + Bx_2^2 + Cx_2 + Dy_2^3 + Ey_2^2 + Fy_2 + G\right) \leq A(\theta x_1 + (1-\theta)x_2)^3 + B(\theta x_1 + (1-\theta)x_2)^2 + C(\theta x_1 + (1-\theta)x_2)$ $+ D(\theta y_1 + (1-\theta)y_2)^3 + E(\theta y_1 + (1-\theta)y_2)^2 + F(\theta y_1 + (1-\theta)y_2) + G$. Rearranging and simplifying, one obtains $(x_1 - x_2)^2((x_1(1+\theta) + x_2(2-\theta))A + B) + (y_1 - y_2)^2((y_1(1+\theta) + y_2(2-\theta))D - B) \leq 0$, which is true since both components of the addition are negative. The first term is negative because $A + B < 0$ and A's coefficient is greater than one, while the second term is negative because $D < 0$, and both D's coefficient and B are positive. It follows that U_P is concave with respect to x and y and that there exists a unique internal maximum (x^*, y^*).

Lower and Upper Bounds of x^*. Since x^* is the unique maximum over the relevant domain, it yields a utility to the founder that is greater than the utility yielded either by $x^* - 1$ or by $x^* + 1$. Formally, we have $U_P(x) \geq U_P(x+1)$ and $U_P(x) \geq U_P(x-1)$. After developing and rearranging terms in each inequality, we obtain a lower and an upper bound for x^* of the form $\underline{x} \leq x \leq \overline{x}$, with $\underline{x} = \frac{1}{4}\frac{\sqrt{\left(a^2\rho^3(N-1)^3 + 32Tc^2(c+\rho(N-1))^2 + 2ca^2\rho^2(N-1)^2\right)}}{a\rho(N-1)\sqrt{2c+\rho(N-1)}} - \frac{1}{4}$, $\overline{x} =$

$\frac{1}{4}\frac{\sqrt{\left(a^2\rho^3(N-1)^3+32Tc^2(c+\rho(N-1))^2+2ca^2\rho^2(N-1)^2\right)}}{a\rho(N-1)\sqrt{2c+\rho(N-1)}}+\frac{3}{4}$. Given that $\underline{x}+1=\overline{x}$ and that x^* is an integer, the value of x^* may fall anywhere in the closed interval $[\underline{x},\overline{x}]$.

Lower and Upper Bounds of y^*. The equilibrium value is what makes the founder indifferent between extending the offer to y^{th} member and earning $V-t^*\left(s_y^{\#}-s_y^{\circ}\right)-T$ and not extending the offer and earning 0. Equating the two outcomes and substituting the corresponding values for transfer and investment into the equation yields the upper bound value $\overline{y}=2\frac{c}{a\rho}\frac{\sqrt{V-T}}{\sqrt{2c+\rho(N-1)}}\frac{c+\rho(N-1)}{N-1}$, and thus the lower bound value $\underline{y}=2\frac{c}{a\rho}\frac{\sqrt{V-T}}{\sqrt{2c+\rho(N-1)}}\frac{c+\rho(N-1)}{N-1}-1$. The value of y^* may fall anywhere in the closed interval $\left[\underline{y},\overline{y}\right]$.

Domain. Since x^* must fall in interval $[1,N]$, we infer the domain of the function for which this result is verified. $\underline{x}\geq 1$ yields condition $T\geq\frac{3}{4}a^2\rho^2(N-1)^2\frac{2c+\rho(N-1)}{c^2(c+\rho(N-1))^2}\equiv\underline{T}$, while $\overline{x}\leq N$ yields condition $T\leq\frac{1}{4}a^2\rho^2(2N-1)(N-1)^3\frac{2c+\rho(N-1)}{c^2(c+\rho(N-1))^2}\equiv\overline{T}$. $\underline{y}\geq 1$ yields $T\leq V-\frac{1}{4}N^2a^2\rho^2(N-1)^2\frac{2c+\rho(N-1)}{c^2(c+\rho(N-1))^2}\equiv\overline{\overline{T}}$, while $\overline{y}\leq N$ yields $T\geq V-a^2\rho^2(N-1)^2\frac{2c+\rho(N-1)}{c^2(c+\rho(N-1))^2}\equiv\underline{\underline{T}}$. Also, $x^*=\begin{cases}N & \text{if} \quad T>\overline{T}\\ 1 & \text{if} \quad T<\underline{T}\end{cases}$ while $y^*=\begin{cases}N & \text{if} \quad T<\underline{\underline{T}}\\ 1 & \text{if} \quad T>\overline{\overline{T}}\end{cases}$. One last condition must be met: $T=\arg\operatorname{solve}\underline{x}\leq\underline{y}\equiv\widehat{T}$. Too long to be reported here, this condition is available from the authors.