Competitive equilibrium

Each member $i \in [1, N]$ chooses her level of $s_i \geq 0$ to maximize her utility $u_i = a_i \sqrt{s_i} - \rho \sum_{j \neq i} s_j - cs_i$, a function that is twice-differentiable and concave. Assuming $\lambda_i \geq 0$ to be the Lagrangian parameters, the optimal level of $s_i$, $s^*_i$, satisfies the necessary and sufficient first-order conditions $a_i \frac{1}{2} s_i^{-\frac{1}{2}} - c + \lambda_i = 0$ and the Kuhn-Tucker conditions $s_i \lambda_i = 0$ for any $i \in [1, N]$, thus forming a system of $2N$ equations and $2N$ variables ($s_i$ and $\lambda_i$). There is no solution possible in which, for any member $i \in [1, N]$, $\lambda_i > 0$, because it would imply $s^*_i = 0$, making the corresponding first-order condition indeterminate. Therefore, the only possible determinate solution has $\lambda_i = 0$ and $s^*_i = \left(\frac{a_i}{2\rho}\right)^2$ for all $i \in [1, N]$.

Social optimum

In any Pareto optimal allocation, the optimal level of $s_i$, $s_i^*$, must maximize the joint surplus of the $N$ members and so must solve $\max_{s_i \geq 0, i \in [1, N]} \sum_{i=1}^{N} (a_i \sqrt{s_i} - cs_i) - \sum_{i=1}^{N} \rho \sum_{j \neq i} s_j$. This problem gives the necessary and sufficient first-order conditions $a_i \frac{1}{2} s_i^{-\frac{1}{2}} - c - (N-1) \rho + \gamma_i = 0$, with $\gamma_i \geq 0$ the Lagrangian parameters, and the Kuhn-Tucker conditions $s_i \gamma_i = 0$ for all $i \in [1, N]$. The problem is solved like the precedent, yielding interior solution $s_i^* = \left(\frac{a_i}{2(c+\rho(N-1))}\right)^2$ for all $i \in [1, N]$.

Solving program $P$

The subsidy rate. We start by determining the optimal subsidy rate, $t^*$. The rate must satisfy two conditions: first, it must be large enough to entice each member to abandon the competitive equilibrium for the social optimum; second, it must be high enough to deter any member from defecting to the competitive equilibrium while holding constant the optimal activity of other members. To meet the first condition, $t$ must make the equilibrium activity under the socially optimal equilibrium at least equal to the equilibrium activity under the competitive equilibrium. Comparing the first-order conditions for each equilibrium (see above), it is straightforward to see that the
condition for the optimal equilibrium is the same as that for the competitive equilibrium minus expression \((N-1)\rho\). Therefore, \(t^* \geq (N-1)\rho\).

To meet the second condition, the incentive constraint in program \(P\) must be met for \(s^*_i = s^*_i\). This means that \(a_i\sqrt{s^*_i} - \rho \sum_{j \neq i} s^*_j - cs^*_i + t (s^*_i - s^*_i) \geq a_i\sqrt{s^*_i} - \rho \sum_{j \neq i} s^*_j - cs^*_i\). Substituting the values of \(s^*_i\) and \(s^*_i\) into the constraint yields \(t^* \geq \frac{c(N-1)\rho}{2\rho + (N-1)\rho}\). Since the right hand side term is smaller than \((N-1)\rho\), it follows that this second condition is not binding, only the first is, and thus \(t^* = (N-1)\rho\).

**Convexity.** To show that program \(P\) is convex with respect to \(x\) and thus has a fixed-point solution, one needs to show that the founder’s utility function, in which we have substituted the values for \(s^*_i\), \(s^*_i\), and \(t^*\), is concave with respect to variables \(x\) and \(y\). Concavity requires that for any pair of distinct points \((x_1, y_1)\) and \((x_2, y_2)\) in the domain of \(U_P\), and for \(0 < \theta < 1\), the following weak inequality holds: 
\[
\theta U_P (x_1, y_1) + (1 - \theta) U_P (x_2, y_2) \leq U_P (\theta (x_1, y_1) + (1 - \theta) (x_2, y_2)).
\]
Developing \(U_P\) and rearranging yields 
\[
U_P = Ax^3 + Bx^2 + Cx + Dy^3 + Ey^2 + Fy + G\text{ with } A = \frac{-1}{6} R, B = \frac{1}{6} R, C = T + \frac{1}{24} R, D = -\frac{1}{12} R, E = -B, F = V - C, G = -T, \text{ and } R = \rho^2 (N-1)^2 a^2 \frac{2c+\rho(N-1)}{c^2(\rho+\rho(N-1))^2}.
\]
This and all subsequent calculations use the functional form for a member’s marginal gain \(a_i = ai\).

Concavity thus requires \(\theta (Ax_1^3 + Bx_1^2 + Cx_1 + Dy_1^3 + Ey_1^2 + Fy_1 + G) + (1 - \theta) (Ax_2^3 + Bx_2^2 + Cx_2 + Dy_2^3 + Ey_2^2 + Fy_2 + G) \leq A (\theta x_1 + (1 - \theta) x_2)^3 + B (\theta x_1 + (1 - \theta) x_2)^2 + C (\theta x_1 + (1 - \theta) x_2) + D (\theta y_1 + (1 - \theta) y_2)^3 + E (\theta y_1 + (1 - \theta) y_2)^2 + F (\theta y_1 + (1 - \theta) y_2) + G\). Rearranging and simplifying, one obtains \(((x_1 - x_2)^2 ((x_1 (1 + \theta) + x_2 (2 - \theta)) A + B) + (y_1 - y_2)^2 ((y_1 (1 + \theta) + y_2 (2 - \theta)) D - B) \leq 0\), which is true since both components of the addition are negative. The first term is negative because \(A + B < 0\) and \(A’s\) coefficient is greater than one, while the second term is negative because \(D < 0\), and both \(D’s\) coefficient and \(B\) are positive. It follows that \(U_P\) is concave with respect to \(x\) and \(y\) and that there exists a unique internal maximum \((x^*, y^*)\).

**Lower and Upper Bounds of \(x^*\).** Since \(x^*\) is the unique maximum over the relevant domain, it yields a utility to the founder that is greater than the utility yielded either by \(x^* - 1\) or by \(x^* + 1\). Formally, we have \(U_P (x) \geq U_P (x+1)\) and \(U_P (x) \geq U_P (x-1)\). After developing and rearranging terms in each inequality, we obtain a lower and an upper bound for \(x^*\) of the form \(\bar{x} \leq x \leq \bar{x}\), with 
\[
\bar{x} = \frac{1}{4} \frac{\sqrt{(a^2\rho^4(N-1)^3 + 327c^2(a+\rho(N-1))^2 + 2c^2\rho^2(N-1)^2)}}{a\rho(N-1)\sqrt{2c+\rho(N-1)}} - \frac{1}{2}, \quad \bar{x} = \frac{1}{4} \frac{\sqrt{(a^2\rho^4(N-1)^3 + 327c^2(a+\rho(N-1))^2 + 2c^2\rho^2(N-1)^2)}}{a\rho(N-1)\sqrt{2c+\rho(N-1)}} + \frac{1}{2}.
\]
Given that \( x + 1 = \bar{x} \) and that \( x^* \) is an integer, the value of \( x^* \) may fall anywhere in the closed interval \([x, \bar{x}]\).

**Lower and Upper Bounds of** \( y^* \). The equilibrium value is what makes the founder indifferent between extending the offer to \( y^{th} \) member and earning \( V - t^* \left( s^y_{\#} - s^y_{\circ} \right) - T \) and not extending the offer and earning 0. Equating the two outcomes and substituting the corresponding values for transfer and investment into the equation yields the upper bound value \( \bar{y} = 2 \frac{c}{ap} \frac{\sqrt{V-T}}{\sqrt{2c+ap(N-1)}} \frac{c+ap(N-1)}{N-1} \), and thus the lower bound value \( y = 2 \frac{c}{ap} \frac{\sqrt{V-T}}{\sqrt{2c+ap(N-1)}} \frac{c+ap(N-1)}{N-1} - 1 \). The value of \( y^* \) may fall anywhere in the closed interval \([y, \bar{y}]\).

**Domain.** Since \( x^* \) must fall in interval \([1, N]\), we infer the domain of the function for which this result is verified. \( x \geq 1 \) yields condition \( T \geq 3 \frac{a^2}{4} \frac{\rho^2}{a}(N-1)^2 \frac{2c+ap(N-1)}{c^2(c+ap(N-1))^2} \equiv T \), while \( \bar{x} \leq N \) yields condition \( T \leq \frac{1}{4} \frac{a^2}{N} \frac{\rho^2}{a^2}(N-1)^2 \frac{2c+ap(N-1)}{c^2(c+ap(N-1))^2} \equiv \bar{T} \). \( y \geq 1 \) yields \( T \leq V - \frac{1}{4} \frac{N^2}{N} \frac{a^2}{N^2} \frac{\rho^2}{a^2}(N-1)^2 \frac{2c+ap(N-1)}{c^2(c+ap(N-1))^2} \equiv \underline{T} \), while \( y \leq N \) yields \( T \geq V - \frac{a^2}{N^2} \frac{\rho^2}{a^2}(N-1)^2 \frac{2c+ap(N-1)}{c^2(c+ap(N-1))^2} \equiv \bar{T} \).

Also, \( x^* = \begin{cases} N & \text{if } T > \bar{T} \\ 1 & \text{if } T < \underline{T} \end{cases} \) while \( y^* = \begin{cases} N & \text{if } T < \bar{T} \\ 1 & \text{if } T > \underline{T} \end{cases} \). One last condition must be met: \( T = \arg \min x \leq y \equiv \hat{T} \). Too long to be reported here, this condition is available from the authors.